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Technical Note

Analytical solution for pulsed laser heating process: convective boundary condition case

B.S. Yilbas [a,*], M. Kalyon [a,b]

[a] ME Department, KFUPM, Dhahran, Saudi Arabia

[b] AE Department, Istanbul Technical University, Istanbul, Turkey

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Abstract

Laser pulse heating offers considerable advantages over the conventional heating methods in industry. In laser industrial applications, in general, an assisting gas is used, which results in convective cooling of the surface during the heating process. Moreover, modelling of the heating process reduces the experimental cost and enhances the understanding of the physical processes involved. In the present study, laser pulse heating of metallic substrates with convective boundary condition at the surface is considered. The time exponentially varying laser pulse is employed in the analysis. A closed form solution pertinent to laser time exponentially varying pulse is obtained using a Laplace transformation method. It is found that analytical solution becomes identical to that obtained previously for a step input pulse intensity when the pulse parameters (β and γ) are set to zero. The effect of Biot number (*Bi*) on the temperature profiles becomes significant as $Bi \geqslant 0.202$. Moreover, pulse parameter (β/γ) has considerable influence on the temperature profiles, in which case, temperature attains low values as β/γ becomes high. © 2002 Published by Elsevier Science Ltd.

Keywords: Exact solution; Laser heating; Convective boundary

1. Introduction

Laser surface treatment of engineering surfaces has several advantages over conventional processes. This is because of low operational cost, precision of operation, and local processing of the surface. Laser surface modification techniques that are used to improve the surface properties of metals include transformation hardening, melting, alloying, cladding, particle injection, etc. The laser surface modification is involved with conduction limited and non-conduction limited heating situations. In the conduction limited heating case, diffusional energy transport dominates the energy transport process while phase change (melting and evaporation) occurs in the non-conduction heating case. One of the applications of conduction limited heating is to increase the temperature in the surface region of the substrate material to austenization level without affecting the bulk temperature of the substrate material. The ensuring self-quenching is rapid enough to eliminate the need for external quenching produced the hard martensite in the heated surface.

Modelling of the heating process minimizes the experimental cost, provides optimization of affecting parameters, and improves the understanding of physical processes involved during the laser workpiece interaction. Considerable research studies were carried out to explore the heating process. Ready [1] introduced an analytical solution to laser pulse heating process. The closed form solution derived was limited to step input pulse and insulated boundary condition at the surface. Analysis of heat conduction in deep penetration welding with a time-modulated laser

* Corresponding author.

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Nomenclature

Bi	Biot number ($h/k\delta$)
C_p	specific heat (J/kg K)
h	heat transfer coefficient (W/m^2 K)
h'	dimensionless heat transfer coefficient ($h/k\delta$)
I	time exponentially varying power intensity (W/m^2)
I_1	power intensity (W/m^2) ($I_0(1-r_f)$)
I_0	laser peak power intensity (W/m^2)
k	thermal conductivity (W/m K)
p	Laplace variable
r_f	reflection coefficient
$T(x,t)$	temperature (K)
$T'(x',\tau)$	dimensionless temperature ($T(x,t)k\delta/I_1$)
T_0	ambient temperature (K)
$\overline{T}$	temperature in Laplace domain (K)
t	time (s)
x	distance (m)
x'	dimensionless distance ($x\delta$)
α	thermal diffusivity (m^2/s)
β	laser pulse parameter (1/s)
β'	dimensionless laser pulse parameter ($=\beta/\alpha\delta^2$)
δ	absorption coefficient (1/m)
γ	Laser pulse parameter for full pulse (1/s)
γ'	dimensionless laser pulse parameter for full pulse ($=\gamma/\alpha\delta^2$)
ρ	density (kg/m^3)
τ	dimensionless time ($\alpha\delta^2 t$)

beam was investigated by Simon et al. [2]. They showed that the time modulated laser beam had little effect on the resulting heat affected zone. Laser heating of a two-layer system was studied by Al-Adawi et al. [4]. They introduced a Laplace integral transformation method when solving the governing equation of heat transfer. They showed that the duration for temperature to reach melting was substrate material thermal properties depended. Hector et al. [3] studied heat conduction in a solid due to a mode-locked laser pulse train. They showed that the differences between the hyperbolic and parabolic models became less extreme as the laser pulse frequency decreased. An analytical solution for laser shortpulse heating of semi-infinite medium was investigated by Yilbas and Sami [5]. The limiting cases for the non-conduction limited heating were investigated and the equation governing the conduction limited heating was deduced. Laser heating for a time exponentially varying pulse intensity was studied by Yilbas [6]. He showed that pulse frequencies of the order of 1 kHz were needed for the integration of the heating process.

In laser heating process, an assisting gas jet is introduced coaxially with a laser beam. The assisting gas jet can be inert or reactive gas depending on the type of heating process required. In laser cutting and welding applications, oxygen gas jet is introduced in the cutting region to enhance the process through a high temperature exothermic reaction while helium or argon gas jet is used when processing non-metallic materials [7]. The assisting gas jet has twofold effects: (i) cooling of the surface, and (ii) either enhancing or preventing the exothermic reaction at the surface, which is irradiated by a laser beam. Shuja and Yilbas [8] introduced the modelling of the convective cooling of the substrate material during laser repetitive pulse heating process. They indicated that the effect of heat transfer coefficient on the surface temperature rise was significant as heat transfer coefficient increased beyond 10^9 W/m^2 K. Blackwell [9] introduced a convective boundary condition at the surface when solving the heat conduction equation due to laser heating pulse. He showed that maximum temperature developed below the surface for heat transfer coefficients higher than 10^8 W/m^2 K. His solution was limited to a step input pulse intensity. However, in most cases, actual laser output pulse intensity varies with time and a mathematical function resembling the temporal variation of the laser pulse shape is necessary to introduce when solving the governing heat transfer equation with convective boundary condition.

In the present study, an analytical approach for laser pulse heating of metallic substrate with convective boundary condition at the surface is considered. A Laplace transformation method is used when solving the governing equation of heat diffusion. A time exponentially varying pulse intensity profile is employed in the analysis.

2. Mathematical modeling

The heat transfer equation for a time exponentially varying laser heating pulse can be written as

$$\frac{\partial^2 T}{\partial x^2} + \frac{I_1 \delta}{k}(e^{-\beta t} - e^{-\gamma t})e^{-\delta x} = \frac{1}{\alpha}\frac{\partial T}{\partial t}. \tag{1}$$

The output pulse from a laser is not easily fitted by a simple mathematical expression, since in general the output from a CO_2 laser is more easily described by approximating the form of the true output by the substraction of two exponential functions. Consequently, the power intensity distribution of time exponentially varying pulse has two exponential terms, which enables it to resemble almost an actual laser pulse [10], i.e.

$$I = I_1(e^{-\beta t} - e^{-\gamma t}), \tag{2}$$

where

$$I_1 = (1 - r_f)I_0,$$

where r_f is the reflection coefficient and I_0 is the peak power intensity, and parameters β and γ can be chosen to give the appropriate rise time for the pulse. Since the governing equation of heat transfer is linear (Eq. (1)), it is unnecessary to solve for the complete pulse (full pulse). Therefore, the complete solution can be obtained by substraction of the solutions for the individual parts of the time exponential pulse (half pulse). It should be noted that for the solution of full pulse, the ambient temperature is considered as zero ($T_0 = 0$). This is necessary since the full pulse solution satisfies the convective boundary condition when $T_0 = 0$. Hence, the heat transfer equation for the half pulse becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{I_1 \delta}{k}e^{-(\beta t + \delta x)} = \frac{1}{\alpha}\frac{\partial T}{\partial t}. \tag{3}$$

The semi-infinite solid body, initially at uniform temperature, with convective boundary condition at the surface is considered. Therefore, the initial and boundary conditions are:

At time $t = 0 \rightarrow T(x,0) = 0$.

At the surface $x = 0 \rightarrow \left[\frac{\partial T}{\partial x}\right]_{x=0} = \frac{h}{k}(T(0,t) - T_0)$ and at $x = \infty \rightarrow T(t,\infty) = 0$.

The solution of Eq. (3) can be obtained possibly through Laplace transformation method, i.e., with respect to t, the Laplace transformation of Eq. (3) yields

$$\frac{\partial^2 \overline{T}}{\partial x^2} + \frac{I_1 \delta}{k}e^{-\delta x}\frac{1}{p+\beta} = \frac{1}{\alpha}\left[p\overline{T} - T(x,0)\right], \tag{4}$$

where $\overline{T} = T(x,p)$ and $T(x,0) = 0$ due to the initial condition. Using the initial condition, Eq. (4) yields

$$\frac{\partial^2 \overline{T}}{\partial x^2} - \frac{p\overline{T}}{\alpha} = -\frac{I_1 \delta}{k}e^{-\delta x}\frac{1}{p+\beta}. \tag{5}$$

Let

$$\lambda^2 = \frac{p}{\alpha} \quad \text{and} \quad H_0 = -\frac{I_1 \delta}{k}\frac{1}{(p+\beta)}.$$

Then it yields

$$\frac{\partial^2 \overline{T}}{\partial x^2} - \lambda^2 \overline{T} = H_0 e^{-\delta x} \tag{6}$$

Eq. (6) has homogeneous ($\overline{T}_h$) and particular solutions ($\overline{T}_p$), i.e.

$$\overline{T} = \overline{T}_h + \overline{T}_p. \tag{7}$$

The homogeneous solution yields

$$\overline{T}_h = C_1 e^{\lambda x} + C_2 e^{-\lambda x}, \tag{8}$$

where C_1 and C_2 are the constants to be determined from the boundary conditions. Similarly the particular solution yields

$$\overline{T}_p = A_0 e^{-\delta x},$$

where A_0 is a constant.

Substituting particular solution into Eq. (6), it yields

$$A_0 \delta^2 e^{-\delta x} - \lambda^2 A_0 e^{-\delta x} = H_0 e^{-\delta x} \tag{9}$$

or

$$A_0 = \frac{H_0}{(\delta^2 - \lambda^2)}.$$

After the rearrangement, the particular solution ($\overline{T}_p$) yields

$$\overline{T}_p = \frac{H_0}{(\delta^2 - \lambda^2)} e^{-\delta x}. \tag{10}$$

Therefore, the solution of Eq. (4) in the Laplace domain becomes

$$\overline{T}(x,p) = C_1 e^{\lambda x} + C_2 e^{-\lambda x} + \frac{H_0}{(\delta^2 - \lambda^2)} e^{-\delta x}. \tag{11}$$

The coefficients in Eq. (11) can be determined from the boundary conditions. Assume $\lambda = \sqrt{p/\alpha} > 0$ and from the boundary condition $T(\infty, t) = 0$, therefore, $C_1 = 0$. Moreover, let $\overline{H}_0 = H_0/(\delta^2 - \lambda^2)$, Eq. (11) yields

$$\overline{T} = \overline{T}(x,p) = C_2 e^{-\lambda x} + \overline{H}_0 e^{-\delta x}. \tag{12}$$

In order to determine C_2, the boundary condition at the surface can be used, i.e.

$$\left[\frac{\partial \overline{T}}{\partial x}\right]_{x=0} = \frac{h}{k}\left[\overline{T}(0,p) - \frac{T_0}{p}\right], \tag{13}$$

where T_0, which is an ambient temperature, is specified. Introducing Eq. (12) into Eq. (13) and knowing from Eq. (12) that $\overline{T}(0,p) = C_2 + \overline{H}_0$, it yields

$$-\lambda C_2 - \delta \overline{H}_0 = \frac{h}{k}\left[C_2 + \overline{H}_0 - \frac{T_0}{p}\right]. \tag{14}$$

Hence, C_2 becomes

$$C_2 = -\frac{\overline{H}_0(h + k\delta)}{(h + k\lambda)} + \frac{T_0 h}{p(h + k\lambda).} \tag{15}$$

Substituting C_2, and the values of H_0, $\overline{H}_0$ and λ into Eq. (12), it yields

$$\overline{T}(x,p) = \frac{I_1 \delta (h + k\delta) e^{-(\sqrt{p/\alpha})x}}{k(p+\beta)(\delta^2 - \frac{p}{\alpha})(h + \frac{k\sqrt{p}}{\sqrt{\alpha}})} + \frac{T_0 h e^{-(\sqrt{p/\alpha})x}}{p(h + \frac{k\sqrt{p}}{\sqrt{\alpha}})} - \frac{I_1 \delta}{k(p+\beta)(\delta^2 - \frac{p}{\alpha})} e^{-\delta x}. \tag{16}$$

The mathematical arrangements of inversion of Eq. (16) is given in Appendix A. The Laplace inversion of Eq. (16) yields

$$T(x,t)=\frac{I_1\alpha\sqrt{\alpha}\delta(h+k\delta)}{k^2}\left\{-\frac{e^{-\delta x}e^{\alpha\delta^2 t}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}-\delta\sqrt{\alpha t}\right)}{2(\beta+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}+\delta\sqrt{\alpha}\right)}-\frac{e^{\delta x}e^{\alpha\delta^2 t}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}+\delta\sqrt{\alpha t}\right)}{2(\beta+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}-\delta\sqrt{\alpha}\right)}\right.$$
$$+\frac{e^{-\beta t}e^{\sqrt{(\beta/\alpha)}xi}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}+\sqrt{\beta t}i\right)}{2(\beta+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}-\sqrt{\beta}i\right)}+\frac{e^{-\beta t}e^{-\sqrt{(\beta/\alpha)}xi}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}-\sqrt{\beta t}i\right)}{2(\beta+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}+\sqrt{\beta}i\right)}$$
$$\left.+\frac{h\sqrt{\alpha}}{k}\frac{e^{hx/k}e^{(h^2/k^2)\alpha t}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h}{k}\sqrt{\alpha t}\right)}{\left(\frac{h^2\alpha}{k^2}+\beta\right)\left(\frac{h^2\alpha}{k^2}-\alpha\delta^2\right)}-\frac{ke^{-\delta x}(e^{-\alpha\delta^2 t}-e^{-\beta t})}{\sqrt{\alpha}(\beta+\alpha\delta^2)(h+k\delta)}\right\}$$
$$+T_0\left[\mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}\right)-e^{hx/k}e^{(h^2/k^2)\alpha t}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h}{k}\sqrt{\alpha t}\right)\right]. \tag{17}$$

The complete analytical solution for the temperature corresponding to the full pulse can be obtained by summing the individual solution corresponding to the half pulse., i.e. the complete solution for the full pulse becomes

$$T(x,t)=\frac{I_1\alpha\sqrt{\alpha}\delta(h+k\delta)}{k^2}\left\{\left[-\frac{e^{-\delta x}e^{\alpha\delta^2 t}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}-\delta\sqrt{\alpha t}\right)}{2(\beta+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}+\delta\sqrt{\alpha}\right)}-\frac{e^{\delta x}e^{\alpha\delta^2 t}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}+\delta\sqrt{\alpha t}\right)}{2(\beta+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}-\delta\sqrt{\alpha}\right)}\right.\right.$$
$$+\frac{e^{-\beta t}e^{(\sqrt{\beta/\alpha})xi}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}+\sqrt{\beta t}i\right)}{2(\beta+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}-\sqrt{\beta}i\right)}+\frac{e^{-\beta t}e^{-\sqrt{(\beta/\alpha)}xi}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}-\sqrt{\beta t}i\right)}{2(\beta+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}+\sqrt{\beta}i\right)}$$
$$\left.+\frac{h\sqrt{\alpha}}{k}\frac{e^{hx/k}e^{(h^2/k^2)\alpha t}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h}{k}\sqrt{\alpha t}\right)}{\left(\frac{h^2\alpha}{k^2}+\beta\right)\left(\frac{h^2\alpha}{k^2}-\alpha\delta^2\right)}-\frac{ke^{-\delta x}(e^{-\alpha\delta^2 t}-e^{-\beta t})}{\sqrt{\alpha}(\beta+\alpha\delta^2)(h+k\delta)}\right]$$
$$-\left[-\frac{e^{-\delta x}e^{\alpha\delta^2 t}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}-\delta\sqrt{\alpha t}\right)}{2(\gamma+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}+\delta\sqrt{\alpha}\right)}-\frac{e^{\delta x}e^{\alpha\delta^2 t}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}+\delta\sqrt{\alpha t}\right)}{2(\gamma+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}-\delta\sqrt{\alpha}\right)}+\frac{e^{-\gamma t}e^{(\sqrt{\gamma/\alpha})xi}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}+\sqrt{\gamma t}i\right)}{2(\gamma+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}-\sqrt{\gamma}i\right)}\right.$$
$$\left.\left.+\frac{e^{-\beta t}e^{-\sqrt{(\gamma/\alpha)}xi}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}-\sqrt{\gamma t}i\right)}{2(\gamma+\alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k}+\sqrt{\gamma}i\right)}+\frac{h\sqrt{\alpha}}{k}\frac{e^{hx/k}e^{(h^2/k^2)\alpha t}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h}{k}\sqrt{\alpha t}\right)}{\left(\frac{h^2\alpha}{k^2}+\gamma\right)\left(\frac{h^2\alpha}{k^2}-\alpha\delta^2\right)}-\frac{ke^{-\delta x}(e^{-\alpha\delta^2 t}-e^{-\gamma t})}{\sqrt{\alpha}(\gamma+\alpha\delta^2)(h+k\delta)}\right]\right\}. \tag{18}$$

Eq. (18) satisfies Eq. (3) and the convective boundary condition for zero ambient temperature ($T_0=0$). In order to make Eq. (18) dimensionless, the following non-dimensional parameters are introduced:

$$\beta'=\frac{\beta}{\alpha\delta^2},\quad \gamma'=\frac{\gamma}{\alpha\delta^2},\quad Bi=h'=\frac{h}{\delta k},\quad \tau=\alpha\delta^2 t,\quad x'=x\delta,\quad T'(x',\tau)=T(x,t)\frac{k\delta}{I_1}.$$

Substituting the above non-dimensional parameters in Eq. (18) and after mathematical arrangement, it yields

$$T'(x',\tau)=(h'+1)\left\{\left[-\frac{e^{-x'}e^{\tau}\ \mathrm{erf}\ c\left(\frac{x'}{2\sqrt{\tau}}-\sqrt{\tau}\right)}{2(\beta'+1)(h'+1)}-\frac{e^{x'}e^{\tau}\ \mathrm{erf}\ c\left(\frac{x'}{2\sqrt{\tau}}+\sqrt{\tau}\right)}{2(\beta'+1)(h'-1)}+\frac{e^{-\beta'}e^{\sqrt{\beta'}x'i}\ \mathrm{erf}\ c\left(\frac{x'}{2\sqrt{\tau}}+\sqrt{\beta'\tau}i\right)}{2(\beta'+1)(h'-\sqrt{\beta'}i)}\right.\right.$$
$$\left.+\frac{e^{-\beta'\tau}e^{-\sqrt{\beta'}x'i}\ \mathrm{erf}\ c\left(\frac{x'}{2\sqrt{\tau}}-\sqrt{\beta'\tau}i\right)}{2(\beta'+1)(h'+\sqrt{\beta'}i)}+\frac{h'e^{h'x'}e^{(h')^2\tau}\ \mathrm{erf}\ c\left(\frac{x'}{2\sqrt{\tau}}+h'\sqrt{\tau}\right)}{((h')^2+\beta')((h')^2-1)}-\frac{e^{-x'}(e^{-\tau}-e^{-\beta'\tau})}{(\beta'+1)(h'+1)}\right]$$
$$-\left[-\frac{e^{-x'}e^{\tau}\ \mathrm{erf}\ c\left(\frac{x'}{2\sqrt{\tau}}-\sqrt{\tau}\right)}{2(\gamma'+1)(h'+1)}-\frac{e^{x'}e^{\tau}\ \mathrm{erf}\ c\left(\frac{x'}{2\sqrt{\tau}}+\sqrt{\tau}\right)}{2(\gamma'+1)(h'-1)}+\frac{e^{-\gamma'}e^{\sqrt{\gamma'}x'i}\ \mathrm{erf}\ c\left(\frac{x'}{2\sqrt{\tau}}+\sqrt{\gamma'\tau}i\right)}{2(\gamma'+1)(h'-\sqrt{\gamma'}i)}\right.$$
$$\left.\left.+\frac{e^{-\gamma'\tau}e^{-\sqrt{\gamma'}x'i}\mathrm{erf}\ c\left(\frac{x'}{2\sqrt{\tau}}-\sqrt{\gamma'\tau}i\right)}{2(\gamma'+1)(h'+\sqrt{\gamma'}i)}+\frac{h'e^{h'x'}e^{(h')^2\tau}\ \mathrm{erf}\ c\left(\frac{x'}{2\sqrt{\tau}}+h'\sqrt{\tau}\right)}{((h')^2+\gamma')((h')^2-1)}-\frac{e^{-x'}(e^{-\tau}-e^{-\gamma'\tau})}{(\gamma'+1)(h'+1)}\right]\right\}. \tag{19}$$

Table 1
Thermal and pulse properties used in the present study

Substrate	δ (1/m) $\times 10^7$	α (m^2/s) $\times 10^{-4}$	C_p (J/kg K)	ρ (kg/m^3)	k (W/m K)	β (1/s) $\times 10^9$	γ (1/s) $\times 10^{10}$	I_0 (W/m^2)
Steel	6.16	0.227	460	7880	80.3	(8.62–17.2)	2.58	10^{10}

Eq. (18) satisfies Eq. (3) and the convective boundary condition for zero ambient temperature ($T_0 = 0$). Eq. (19) is used to compute the dimensionless surface temperature profiles at the surface for the full laser heating pulse. The substrate thermal and the pulse properties employed in the computation are given in Table 1.

3. Results and discussion

The analytical solution for the laser pulse heating process including the convective boundary condition at the surface is obtained for time exponentially varying pulse intensity. The closed form solution is derived using a Laplace transformation method. The time exponentially varying pulse is presented in terms of exponential functions. The closed form solution reduces to the analytical solution obtained previously [9] for a step input pulse intensity when the pulse parameters are set to zero ($\beta = 0$ and $\gamma = 0$). The analysis related to step input pulse is given in Appendix B. The dimensionless power intensity distribution employed in the analysis is shown in Fig. 1 for two β/γ values.

Fig. 2 shows the dimensionless temperature variation with dimensionless distance for $Bi = 0.202$ and $\beta/\gamma = 0.32$ as dimensionless time is variable. Temperature in the surface vicinity of the substrate material attains almost the same gradient ($\partial T'/\partial x'$) for heating period $4.29 \leqslant \tau \leqslant 8.6$. As the heating period progresses, the positive temperature gradient occurs in the surface region, i.e. temperature attains higher values below the surface as compared to its counterpart at the surface. This is because of the convective cooling of the surface, since Bi is high ($Bi = 0.202$). In this case, contribution of convective cooling to energy transport process is considerably high. This is more pronounced as the heating progresses, provided that intensity of the laser output pulse reduces as $\tau > 7$, which can be seen from Fig. 1. Therefore, as the beam intensity reduces, the increase in internal energy reduces, which in turn suppresses the temperature rise in the surface region. Although Bi is kept constant in Fig. 2, the amount of heat convected is high despite the low surface temperature. Temperature attains lower values inside the substrate material as $\tau > 8.6$. This is because of less laser energy being absorbed by the substrate material due to time varying laser pulse.

Fig. 3 shows the dimensionless temperature distribution inside the substrate material for two Bi and $\beta/\gamma = 0.31$ as dimensionless time is variable. Temperature profiles corresponding to $Bi = 0.202 \times 10^{-2}$ and 0.202×10^{-5} coincide for all the heating periods. This indicates that the effect of convective cooling of the surface on the temperature profiles is negligibly small for the Bi values, i.e. the influence of convective cooling of the surface on the energy transport mechanism taking place in the surface region is negligible. This can also be seen from Fig. 4, in which surface temperature with Bi is shown. The surface temperature remains same up to $Bi > 0.15 \times 10^{-2}$. As Bi increases further, surface temperature drops sharply. This is because of convective cooling of the surface. The maximum temperature

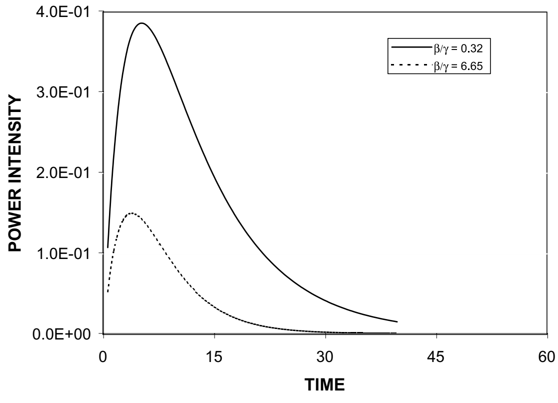

Fig. 1. Dimensionless power intensity distribution with time at two β/γ.

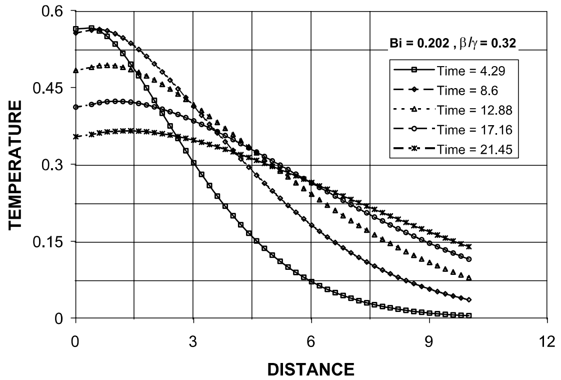

Fig. 2. Dimensionless temperature distribution with dimensionless distance inside substrate material at different dimensionless heating periods for $Bi = 0.202$ and $\beta/\gamma = 0.32$.

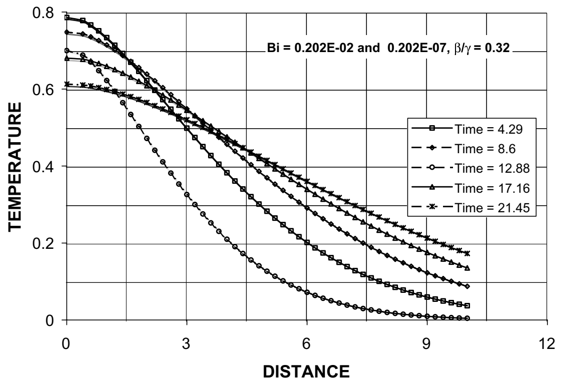

Fig. 3. Dimensionless temperature distribution with dimensionless distance inside substrate material at different dimensionless heating periods for $Bi = 0.202\mathrm{E}-02$ and $0.202\mathrm{E}-07$, and $\beta/\gamma = 0.32$.

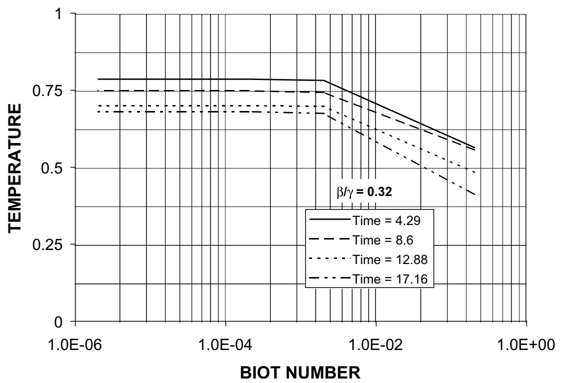

Fig. 4. Dimensionless surface temperature distribution with Biot number at different dimensionless heating periods for $\beta/\gamma = 0.32$.

occurs at the surface for small *Bi* values and the temperature gradient reduces as the dimensionless distance increases towards the solid bulk, provided that the temperature gradient in the surface vicinity is smaller as compared to that corresponding to next to surface vicinity ($x' = 0.5$). This is more pronounced in the early heating period ($\tau = 8.6$). In this case, energy content in the laser pulse is high (Fig. 1). This results in increasing internal energy gain of the substrate material in the surface vicinity. Although the diffusional heat transfer enhances due to temperature gradient in this region, the internal energy gain dominates the diffusional heat transfer from the surface region to the bulk of the substrate material. Consequently, temperature gradient remains low in the surface vicinity. As the distance from the surface increases beyond the absorption depth, the temperature gradient drops sharply, in which case, diffusional heat transport dominates the energy transport mechanism.

Fig. 5 shows the dimensionless temperature distribution along the dimensionless distance for two pulse parameters (β/γ). The effect of β/γ on the temperature profiles is considerable. In this case, large β/γ value results in low temperature in the surface vicinity of the substrate material. This is because of the power intensity distribution (Fig. 1), i.e. large β/γ value results in low power intensity. Moreover, temperature gradient corresponding to high β/γ value is lower than that corresponding to low β/γ value. In this case, the internal energy gain of the substrate material in the surface region is low for high β/γ value. Consequently, diffusional energy transport becomes less from surface region to the bulk of the substrate material.

Figs. 6 and 7 show the maximum temperature inside the substrate material for $Bi = 0.202$ for two β/γ values and location of maximum temperature with time, respectively. Maximum temperature moves away from the surface as the heating period progresses. The magnitude of maximum temperature reduces for high value of β/γ. The location of maximum temperature varies almost linearly with increasing period (Fig. 7) for $\beta/\gamma = 0.32$. However, this behavior changes as $\beta/\gamma = 0.66$. In this case, the location of maximum temperature inside the substrate material remains same as the heating period progresses ($\tau > 13$). This indicates that surface temperature becomes low as the heating period progresses because of the less power intensity in the heating pulse. Although Bi is same for both β/γ values, low surface temperature lowers the convective energy loss from the surface. This prevents the location of maximum temperature to move towards the solid bulk of the substrate material.

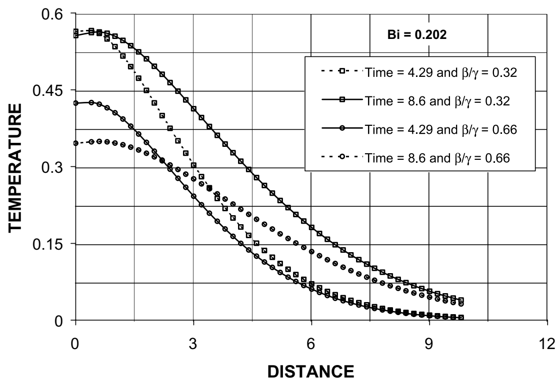

Fig. 5. Dimensionless temperature distribution with dimensionless distance inside susbtrate material at two dismensionless heating periods for two β/γ (b/g) and $Bi = 0.202$.

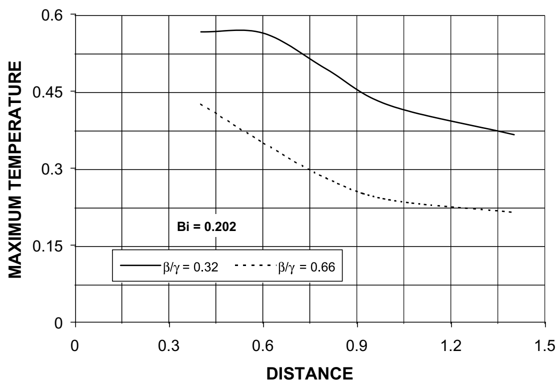

Fig. 6. Maximum temperature with dimensionless distance for two β/γ.

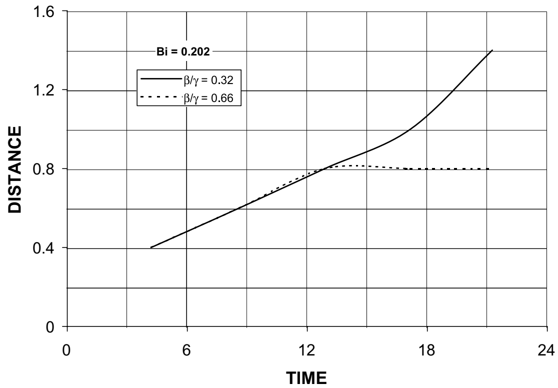

Fig. 7. Dimensionless distance and time at which maximum temperature occurs for two β/γ.

4. Conclusions

Laser pulse heating with convective boundary condition at the surface is considered. The time exponentially varying laser pulse is employed in the analysis. A closed form solution for temperature distribution is obtained using a Laplace transformation method. It is found that the closed form solution becomes identical to the solution obtained previously for a laser step input pulse. The effect of *Bi* on the temperature distribution becomes significant as $Bi > 0.202$. The specific conclusions derived from the present work can be listed as follows:

1. In the early heating period, temperature gradient drops sharply in the region next to the surface vicinity ($x' > 1.3$). In this case, the power intensity in the pulse profile is higher in the early heating period due to time exponentially varying pulse and the diffusional energy transfer from the surface vicinity to the bulk of the substrate material dominates over the energy transport mechanism in this region. As *Bi* reduces beyond 0.202×10^{-2}, the effect of convective cooling on the temperature profiles becomes negligibly small.
2. The influence of pulse parameter (β/γ) on temperature profiles is considerable. In this case, increasing β/γ reduces temperature inside the substrate material. This is because of the power intensity distribution in the heating pulse, i.e. the magnitude of power intensity reduces as β/γ increases. The temperature gradient inside the substrate material changes as β/γ varies, which is more pronounced as the heating period progresses.
3. Maximum temperature moves away from the surface as the heating period progresses, provided that the magnitude of maximum temperature is high in the early heating period. The effect of β/γ on the location of maximum temperature inside the substrate material is significant, i.e. the location of maximum temperature increases almost linearly with time for $\beta/\gamma = 0.32$ while the location increases up to the heating period of $\tau = 13$, and as the heating period increases further, the location of maximum temperature remains constant inside the substrate material.

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Appendix A. Laplace inversion of temperature equation

The solution of temperature distribution in the Laplace domain (Eq. (16)) is

$$\overline{T} = \frac{I_1\delta(h+k\delta)\mathrm{e}^{-(\sqrt{p/\alpha})x}}{k(p+\beta)\left(\delta^2-\frac{p}{\alpha}\right)\left(h+\frac{k\sqrt{p}}{\sqrt{\alpha}}\right)} + \frac{T_0 h\mathrm{e}^{-(\sqrt{p/\alpha})x}}{p\left(h+\frac{k\sqrt{p}}{\sqrt{\alpha}}\right)} - \frac{I_1\delta}{k(p+\beta)\left(\delta^2-\frac{p}{\alpha}\right)}\mathrm{e}^{-\delta x}. \tag{A.1}$$

Let us define the following terms:

$$\text{Term1} = \frac{I_1\delta(h+k\delta)}{k(p+\beta)\left(\delta^2-\frac{p}{\alpha}\right)\left(h+\frac{k\sqrt{p}}{\sqrt{\alpha}}\right)},\quad \text{Term2} = -\frac{I_1\delta}{k(p+\beta)\left(\delta^2-\frac{p}{\alpha}\right)},\quad \text{Term3} = \frac{T_0 h}{p\left(h+\frac{k\sqrt{p}}{\sqrt{\alpha}}\right)}.$$

Let us introduce the following parameters:

$$a_{10} = -\frac{I_1\alpha\sqrt{\alpha}\delta(h+k\delta)}{k^2},\quad a_{20} = \frac{I_1\alpha\delta}{k},\quad a_{30} = \frac{h\sqrt{\alpha}}{k}T_0,\quad w_1 = \frac{h\sqrt{\alpha}}{k}.$$

It should be noted that a_{10}, a_{20}, a_{30} are introduced for Term1, Term2, and Term3, respectively. Then, Eq. (A.1) becomes

$$\overline{T}(x,p) = \frac{a_{10}e^{-(\sqrt{p/\alpha})x}}{(p+\beta)(p-\alpha\delta^2)(\sqrt{p}+w_1)} + \frac{a_{20}e^{-\delta x}}{(p+\beta)(p-\alpha\delta^2)} + \frac{a_{30}e^{-(\sqrt{p/\alpha})x}}{p(\sqrt{p}+w_1)}. \tag{A.2}$$

However, one can write for Term1

$$\frac{1}{(p+\beta)(p-\alpha\delta^2)(\sqrt{p}+w_1)} = \frac{1}{(\sqrt{p}+i\sqrt{\beta})(\sqrt{p}-i\sqrt{\beta})(\sqrt{p}-\delta\sqrt{\alpha})(\sqrt{p}+\delta\sqrt{\alpha})(\sqrt{p}+w_1)}$$

or

$$\frac{1}{(p+\beta)(p-\alpha\delta^2)(\sqrt{p}+w_1)} = \frac{l_1}{\sqrt{p}+i\sqrt{\beta}} + \frac{l_2}{\sqrt{p}-i\sqrt{\beta}} + \frac{l_3}{\sqrt{p}-\delta\sqrt{\alpha}} + \frac{l_4}{\sqrt{p}+\delta\sqrt{\alpha}} + \frac{l_5}{\sqrt{p}+w_1},$$

where the residue values of l_1, l_2, l_3, l_4, and l_5 are obtained as

$$l_1 = \frac{1}{2\sqrt{\beta}i(\beta+\alpha\delta^2)(-\sqrt{\beta}i+w_1)},\quad l_2 = -\frac{1}{2\sqrt{\beta}i(\beta+\alpha\delta^2)(\sqrt{\beta}i+w_1)},\quad l_3 = \frac{1}{2\sqrt{\alpha}\delta(\beta+\alpha\delta^2)(\delta\sqrt{\alpha}+w_1)},$$
$$l_4 = -\frac{1}{2\sqrt{\alpha}\delta(\beta+\alpha\delta^2)(-\delta\sqrt{\alpha}+w_1)},\quad l_5 = \frac{1}{(w_1^2+\beta)(w_1^2-\alpha\delta^2)}.$$

In order to obtain $£^{-1}$Term1, we use the following formula from the Laplace table [11]:

$$£^{-1}\frac{e^{-m\sqrt{p}}}{n+\sqrt{p}} = \frac{1}{\sqrt{\pi t}}e^{-m^2/4t} - ne^{mn}e^{n^2t}\,\text{erf}\,c\left(n\sqrt{t}.+\frac{m}{2\sqrt{t}}\right),$$

where $m \geqslant 0$. Therefore, $£^{-1}$Term1 can be written as

$$\begin{aligned}
£^{-1}\text{Term1} = a_{10}\Bigg\{ & l_1\left[\frac{1}{\sqrt{\pi t}}e^{-x^2/4\alpha t} - \sqrt{\beta}ie^{\sqrt{\beta}(xi/\sqrt{\alpha})}e^{-\beta t}\,\text{erf}\,c\left(\sqrt{\beta t}i+\frac{x}{2\sqrt{\alpha t}}\right)\right] \\
& + l_2\left[\frac{1}{\sqrt{\pi t}}e^{-x^2/4\alpha t} + \sqrt{\beta}ie^{-\sqrt{\beta}(xi/\sqrt{\alpha})}e^{-\beta t}\,\text{erf}\,c\left(-\sqrt{\beta t}i+\frac{x}{2\sqrt{\alpha t}}\right)\right] + l_3\left[\frac{1}{\sqrt{\pi t}}e^{-x^2/4\alpha t}\right. \\
& \left.+\delta\sqrt{\alpha}e^{-\delta x}e^{\alpha\delta^2 t}\,\text{erf}\,c\left(-\delta\sqrt{\alpha t}+\frac{x}{2\sqrt{\alpha t}}\right)\right] + l_4\left[\frac{1}{\sqrt{\pi t}}e^{-x^2/4\alpha t} - \delta\sqrt{\alpha}e^{\delta x}e^{\alpha\delta^2 t}\,\text{erf}\,c\left(\delta\sqrt{\alpha t}+\frac{x}{2\sqrt{\alpha t}}\right)\right] \\
& + l_5\left[\frac{1}{\sqrt{\pi t}}e^{-x^2/4\alpha t} - w_1e^{w_1(x/\sqrt{\alpha})}e^{w_1^2 t}\,\text{erf}\,c\left(w_1\sqrt{t}+\frac{x}{2\sqrt{\alpha t}}\right)\right]\Bigg\}.
\end{aligned} \tag{A.3}$$

After the arrangement of Eq. (A.3), $£^{-1}$Term1 becomes

$$\begin{aligned}
£^{-1}\text{Term1} = a_{10}\Bigg\{ & \sqrt{\beta}ie^{-\beta t}\left[-l_1e^{(\sqrt{\beta/\alpha})ix}\,\text{erf}\,c\left(\sqrt{\beta t}i+\frac{x}{2\sqrt{\alpha t}}\right) + l_2e^{-(\sqrt{\beta/\alpha})ix}\,\text{erf}\,c\left(-\sqrt{\beta t}i+\frac{x}{2\sqrt{\alpha t}}\right)\right] \\
& + \delta\sqrt{\alpha}e^{\alpha\delta^2 t}\left[l_3e^{-\delta x}\,\text{erf}\,c\left(-\delta\sqrt{\alpha t}+\frac{x}{2\sqrt{\alpha t}}\right) - l_4e^{\delta x}\,\text{erf}\,c\left(\delta\sqrt{\alpha t}+\frac{x}{2\sqrt{\alpha t}}\right)\right] \\
& - w_1l_5e^{w_1(x/\sqrt{\alpha})}e^{w_1^2 t}\,\text{erf}\,c\left(w_1\sqrt{t}+\frac{x}{2\sqrt{\alpha t}}\right)\Bigg\},
\end{aligned} \tag{A.4}$$

where

$$l_1 + l_2 + l_3 + l_4 + l_5 = 0.$$

The Laplace inversion of Term2 can also be obtained as

$$£^{-1}\text{Term2} = a_{20}\mathrm{e}^{-\delta x}\left[-\frac{1}{\beta+\alpha\delta^2}\mathrm{e}^{-\beta t} + \frac{1}{\beta+\alpha\delta^2}\mathrm{e}^{\alpha\delta^2 t}\right]$$

or

$$£^{-1}\text{Term2} = \frac{a_{20}}{\beta+\alpha\delta^2}(\mathrm{e}^{\alpha\delta^2 t} - \mathrm{e}^{-\beta t})\mathrm{e}^{-\delta x}. \tag{A.5}$$

The following formula from the Laplace table [11] can be used to obtain the inversion of Term3 ($£^{-1}$Term3), i.e.

$$£^{-1}\frac{n\mathrm{e}^{-m\sqrt{p}}}{p(\sqrt{p}+n)} = -\mathrm{e}^{mn}\mathrm{e}^{n^2 t}\ \mathrm{erf}\ c\left(n\sqrt{t} + \frac{m}{2\sqrt{t}}\right) +\ \mathrm{erf}\ c\left(\frac{m}{2\sqrt{t}}\right). \tag{A.6}$$

Therefore, the inversion of Term3 becomes

$$£^{-1}\text{Term3} = \frac{a_{30}}{w_1}\left[-\mathrm{e}^{(w_1/\sqrt{\alpha})x}\mathrm{e}^{w_1^2 t}\ \mathrm{erf}\ c\left(w_1\sqrt{t} + \frac{x}{2\sqrt{\alpha t}}\right) + \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]. \tag{A.7}$$

After substituting the Laplace inversion form of terms in Eq. (19), the solution yields

$$\begin{aligned} T(x,t) = a_{10}\Bigg\{ & \sqrt{\beta}i\mathrm{e}^{-\beta t}\left[-l_1\mathrm{e}^{(\sqrt{\beta/\alpha})ix}\ \mathrm{erf}\ c\left(\sqrt{\beta t i} + \frac{x}{2\sqrt{\alpha t}}\right) + l_2\mathrm{e}^{-(\sqrt{\beta/\alpha})ix}\ \mathrm{erf}\ c\left(-\sqrt{\beta t i} + \frac{x}{2\sqrt{\alpha t}}\right)\right] \\ & + \delta\sqrt{\alpha}\mathrm{e}^{\alpha\delta^2 t}\left[l_3\mathrm{e}^{-\delta x}\ \mathrm{erf}\ c\left(-\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) - l_4\mathrm{e}^{\delta x}\ \mathrm{erf}\ c\left(\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right)\right] \\ & - w_1 l_5\mathrm{e}^{w_1(x/\sqrt{\alpha})}\mathrm{e}^{w_1^2 t}\ \mathrm{erf}\ c\left(w_1\sqrt{t} + \frac{x}{2\sqrt{\alpha t}}\right)\Bigg\} + \frac{a_{20}}{(\beta+\alpha\delta^2)}(\mathrm{e}^{\alpha\delta^2 t} - \mathrm{e}^{-\beta t})\mathrm{e}^{-\delta x} \\ & + \frac{a_{30}}{w_1}\left[-\mathrm{e}^{w_1(x/\sqrt{\alpha})}\mathrm{e}^{w_1^2 t}\ \mathrm{erf}\ c\left(w_1\sqrt{t} + \frac{x}{2\sqrt{\alpha t}}\right) + \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]. \end{aligned} \tag{A.8}$$

Appendix B. Convergence to step input pulse intensity

To deduce the closed-form solution derived from the present analysis to other analytical solutions reported in the literature [9], the step input intensity pulse with convective boundary condition at the surface is considered. The analytical solution for the step input intensity pulse with convective boundary condition was obtained previously by Blackwell [9]. In order to introduce the step input intensity pulse in the closed-form solution derived at present, the pulse parameter (β) in the source term of Eq. (3) or in the closed-form solution (Eq. (17)) should be set to zero. Moreover, the initial and ambient temperatures, and the symbol of absorption coefficient used in Blackwell's solution are different than those employed in the present analysis, i.e. the initial temperature is set to zero, the ambient temperature is denoted as T_0, and δ is used for the absorption coefficient in the present study. To achieve this the following steps are considered:

$$T(x,t) = £^{-1}[\text{Term1}]_{\beta=0} + \frac{a_{20}}{\alpha\delta^2}(\mathrm{e}^{\alpha\delta^2 t} - 1)\mathrm{e}^{-\delta x} + T_0\left[-\mathrm{e}^{w_1(x/\sqrt{\alpha})}\mathrm{e}^{w_1^2 t}\ \mathrm{erf}\ c\left(w_1\sqrt{t} + \frac{x}{2\sqrt{\alpha t}}\right) +\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}\right)\right], \tag{B.1}$$

where

$$\begin{aligned} £^{-1}[\text{Term1}]_{\beta=0} = a_{10}\Bigg\{ & \left[-\frac{1}{2\alpha\delta^2 w_1}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}\right) - \frac{1}{2\alpha\delta^2 w_1}\ \mathrm{erf}\ c\left(\frac{x}{2\sqrt{\alpha t}}\right)\right] \\ & \times\delta\sqrt{\alpha}\mathrm{e}^{\alpha\delta^2 t}\left[l_3\mathrm{e}^{-\delta x}\ \mathrm{erf}\ c\left(-\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) - l_4\mathrm{e}^{\delta x}\ \mathrm{erf}\ c\left(\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right)\right]\Bigg\} \\ & - w_1 l_5\mathrm{e}^{w_1(x/\sqrt{\alpha})}\mathrm{e}^{w_1^2 t}\ \mathrm{erf}\ c\left(w_1\sqrt{t} + \frac{x}{2\sqrt{\alpha t}}\right). \end{aligned} \tag{B.2}$$

Recall that for $\beta = 0, l_3, l_4$, and l_5 become

$$l_3 = \frac{1}{2\alpha\sqrt{\alpha}\delta^3(\delta\sqrt{\alpha}+w_1)}, \quad l_4 = -\frac{1}{2\alpha\sqrt{\alpha}\delta^3(-\delta\sqrt{\alpha}+w_1)}, \quad l_5 = \frac{1}{w_1^2(w_1^2-\alpha\delta^2)}, \quad a_{10} = \frac{I_1\alpha\sqrt{\alpha}\delta(h+k\delta)}{k^2},$$
$$a_{20} = \frac{I_1\alpha\delta}{k}, \quad w_1 = \frac{h\sqrt{\alpha}}{k}.$$

Substituting $l_3, l_4, l_5, a_{10}, a_{20}$, and w_1 in Eq. (B.1), and after rearranging it yields

$$\begin{aligned} T(x,t) = {} & T_0\left[\text{erf }c\left(\frac{x}{2\sqrt{\alpha t}}\right) - \mathrm{e}^{\left((h^2/k^2)\alpha t+(h/k)x\right)}\text{ erf }c\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h}{k}\sqrt{\alpha t}\right)\right] + \frac{I_1}{k\delta}\left\{\left(1+\frac{k\delta}{h}\right)\text{ erf }c\left(\frac{x}{2\sqrt{\alpha t}}\right)\right. \\ & -\frac{1}{2}\mathrm{e}^{(\alpha\delta^2 t-\delta x)}\text{ erf }c\left(-\delta\sqrt{\alpha t}+\frac{x}{2\sqrt{\alpha t}}\right) - \frac{1}{2}\frac{\left(\frac{h}{k\delta}+1\right)}{\left(\frac{h}{k\delta}-1\right)}\mathrm{e}^{(\alpha\delta^2 t+\delta x)}\text{ erf }c\left(\delta\sqrt{\alpha t}+\frac{x}{2\sqrt{\alpha t}}\right) \\ & \left. + \frac{1}{\frac{h}{k\delta}\left(\frac{h}{k\delta}-1\right)}\mathrm{e}^{\left((h/k)x+(h^2/k^2)\alpha t\right)}\text{ erf }c\left(\frac{h}{k}\sqrt{\alpha t}+\frac{x}{2\sqrt{\alpha t}}\right) + \mathrm{e}^{-\delta x}(\mathrm{e}^{\alpha\delta^2 t}-1)\right\}, \end{aligned} \tag{B.3}$$

which is exactly the same as Blackwell's solution for a step input intensity pulse.

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